Pure Gauge Configurations and Solutions to Fermionic Superstring Field Theories Equations of Motion

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Abstract

Recent results on solutions to the equation of motion of the cubic fermionic string field theory and an equivalence of non-polynomial and cubic string field theory are discussed. To have a possibility to deal with both $GSO(+)$ and $GSO(-)$ sectors in the uniform way a matrix formulation for the NS fermionic SFT is used. In constructions of analytical solutions to open string field theories truncated pure gauge configurations parameterized by wedge states play an essential role. The matrix form of this parametrization for the NS fermionic SFT is presented. Using the cubic open superstring field theory as an example we demonstrate explicitly that for the large parameter of the perturbation expansion these truncated pure gauge configurations give divergent contributions to the equation of motion on the subspace of the wedge states. The perturbation expansion is cured by adding extra terms that are nothing but the terms necessary for the equation of motion contracted with the solution itself to be satisfied.

This work is dedicated to the memory of Aliosha Zamolodchikov

1 Introduction

It is well known that string field theories (SFT) describe infinite number of local fields. Just by this reason finding nontrivial solutions to classical SFT is a rather nontrivial problem. This is a reason why the Schnabl construction of the tachyon solution [1] in the Witten open bosonic SFT [2] attend a lot of attentions [3] - [22]. It turns out that the tachyon solution is closely related to pure gauge solutions. More precisely, Schnabl's solution is a regularization of a singular limit of a pure gauge configuration [1, 3]. The presence of pure gauge solutions in the bosonic SFT is related to the Chern-Simons form of the Witten cubic action. The Schnabl solution is distinguished by the fact that it describes a true vacuum of SFT, i.e. a vacuum on which the Sen conjecture is realized. Since the pure gauge solutions do not shift the vacuum energy the correct shift of the vacuum energy by the Schnabl solution is rather non-trivial fact and its deep origin is still unclear for us.

The purpose of this report is to present recent results concerning the generalization of the Schnabl solution to the fermionic case.

It is natural to expect that a solution being a singular limit of a pure gauge solution also exist in the cubic super SFT (SSFT) [23, 25]. But for the superstring case there is no a priori a reason to deal with the Sen conjecture, since the perturbative vacuum is stable (there is no tachyon). However a nontrivial (not pure gauge) solution to the SSFT equation of motion does exist [33]. The physical meaning of this solution is still unclear. It may happen that it is related with a spontaneous supersymmetry breaking (compare with [24]).

There is also a non-polynomial formulation of the SSFT [31]. A solution of equation of motion for marginal deformations in the non-polynomial SSFT has been obtained in [35, 36]. This construction became clear after realization an explicit relation between solutions to the cubic and non-polynomial superstring field theories [32]. These theories include only the $GSO(+)$ sector of the NS string. There are also two versions of the NS fermionic SFT that includes both $GSO(+)$ and $GSO(-)$ sectors, cubic [27] and non-polynomial [30]. Just the NS fermionic SFT with two sectors is used to describe non-BPS branes. The Sen conjecture has been checked by the level truncation for the non-polynomial and cubic cases in [34] and [27], respectively. A solution to the equation of motion of the cubic SFT describing the NS string with both $GSO(+)$ and $GSO(-)$ sectors has been constructed in [28]. On this solution the Sen conjectures takes place.

To make a construction of the solution [28] more clear it is useful to incorporate a matrix version of NS fermionic SFT with $GSO(+)$ and $GSO(-)$ [29]. In the matrix formulation an explicit relation between solutions to the cubic and non-polynomial theories become more clear and it gives an explicit formula for solutions to the non-polynomial theory [30] via solutions [28] to ABKM theory [27].

The Schnabl solution Ψ consists of two pieces and is defined by the limit:

$$\Psi = \lim_{N\to\infty} [\, \Psi_N(1) - \psi_N \,] \,, \quad \Psi_N(1) = \sum_{n=0}^{N} \psi'_n, \tag{1}$$

where the states ψ'_n defined for any real $n \geq 0$ are made of the wedge state [37, 38, 39].

It was shown [1] that the string field Ψ in (1) solves the equation of motion of Witten's SFT contracted with any state C in the Fock space with a finite number of string excitations.

$$\langle C, Q\Psi + \Psi \star \Psi \rangle = 0. \tag{2}$$

On the other hand to check the Sen conjecture, one has to use the equation of motion contracted with a solution itself. The ψ_N piece in (1) is necessary for the equation of motion contracted with the solution itself to be satisfied [3, 32].

We note that the pure gauge part of the Erler configuration does not solve the equation of motion contracted with wedge states ψ_m

$$\langle \psi_m, Q\Psi_\infty(1) + \Psi_\infty(1) \star \Psi_\infty(1) \rangle \neq 0. \tag{3}$$

It is possible to add extra terms ψ_N to $\Psi_\infty(1)$ to get a solution in the sense of (3). These are just the terms that have been used previously to get a desirable value of the action [33].

The paper is organized as follows. In Section 2 a matrix formulation for the NS fermionic SFT is presented. In Section 3 we contribute to a discussion [32] of the classical equivalence of the non-polynomial theory of Berkovits with $GSO(\pm)$ sectors [30] (here we refer to this theory as the Berkovits, Sen and Zwiebach theory) and the cubic theory of Belov, Koshelev and two of us [27]. In Section 4 perturbative parameterizations of special pure gauge configurations are presented. These pure gauge configurations are used in the Erler superstring field theory solution [33] and in the tachyon fermion solution [28]. We demonstrate that $\lambda = 1$ limit of these pure gauge solutions is in fact a singular point and we use a simple prescription to cure divergences. We show that this prescription gives the same answer as the requirement of validity of the equation of motion contracted with the solution itself.

2 Pure Gauge Configurations in Cubic SFT for Fermion String with $GSO(+)$ and $GSO(-)$ Sectors in Matrix Notations

2.1 Cubic Fermion SFT with $GSO(-)$ Sector in Matrix Notations

The action for covariant superstring field theory with $GSO(+)$ and $GSO(-)$ sectors was proposed at [27]:

$$\begin{aligned} S[\Phi_+, \Phi_-] \quad = \quad & -\frac{1}{g_0^2}\left[\frac{1}{2}\langle Y_{-2}\Phi_+, Q\Phi_+\rangle + \frac{1}{3}\langle Y_{-2}\Phi_+, \Phi_+ \star \Phi_+\rangle \right. \\ & \left. + \frac{1}{2}\langle Y_{-2}\Phi_-, Q\Phi_-\rangle - \langle Y_{-2}\Phi_+, \Phi_- \star \Phi_-\rangle \right]. \end{aligned} \tag{4}$$

The equations of motion read ($\star$ stands for Witten's string field product)

$$Q\Phi_+ + \Phi_+ \star \Phi_+ - \Phi_- \star \Phi_- \quad = \quad 0, \tag{5}$$

$$Q\Phi_- + \Phi_+ \star \Phi_- - \Phi_- \star \Phi_+ \quad = \quad 0. \tag{6}$$

The string fields Φ_+ and Φ_- have definite and opposite Grassman parity, to be fixed below. The parity $|\Phi|$ leads to the Leibniz rule

$$Q(\Phi \star \Psi) = Q\Phi \star \Psi + (-)^{|\Phi|}\Phi \star Q\Psi. \tag{7}$$

It is useful to introduce matrix notations [29] by tensoric string fields and operators with appropriate 2×2 matrices. In this notations the action (4) reads

$$S[\widehat{\Phi}] = -\frac{1}{g_0^2}\left[\frac{1}{2}\langle \widehat{Y}_{-2}\widehat{\Phi}, \widehat{Q}\widehat{\Phi}\rangle + \frac{1}{3}\langle \widehat{Y}_{-2}\widehat{\Phi}, \widehat{\Phi} \star \widehat{\Phi}\rangle\right], \tag{8}$$

the string field $\widehat{\Phi}$ is given by [29]

$$\widehat{\Phi} = \Phi_+ \otimes \sigma_3 + \Phi_- \otimes i\sigma_2, \tag{9}$$

and

$$\widehat{Q} = Q \otimes \sigma_3, \quad \widehat{Y}_{-2} = Y_{-2} \otimes \sigma_3, \tag{10}$$

σ_i are Pauli matrices, Q is the BRST charge and Y_{-2} is a double-step picture changing operator.

The parity assignment and σ_i algebra lead to the Leibniz rule

$$\widehat{Q}(\widehat{\Phi} \star \widehat{\Psi}) = (\widehat{Q}\widehat{\Phi}) \star \widehat{\Psi} + (-)^{|\widehat{\Phi}|}\widehat{\Phi} \star (\widehat{Q}\widehat{\Psi}), \tag{11}$$

where

$$|\widehat{\Phi}| \equiv |\Phi_+|. \tag{12}$$

The action is only nonvanishing for a string field of degree 1. We also have

$$\begin{aligned} |\widehat{\Phi} \star \widehat{\Psi}| &= |\widehat{\Phi}| + |\widehat{\Psi}|, & (13)\\ |\widehat{Q}\widehat{\Phi}| &= 1 + |\widehat{\Phi}|, & (14)\\ |\widehat{Q}| &= 1, & (15)\\ |\widehat{\Phi}| &= 1. & (16) \end{aligned}$$

The equations of motion (5) in the matrix notations read

$$\widehat{Q}\widehat{\Phi} + \widehat{\Phi} \star \widehat{\Phi} = 0. \tag{17}$$

2.2 Pure Gauge Solutions to Equation of Motion

Pure gauge solutions to (17) have the form

$$\widehat{\Phi} = \widehat{\Omega}^{-1} \star \widehat{Q}\widehat{\Omega} = -\widehat{Q}\widehat{\Omega} \star \widehat{\Omega}^{-1} \tag{18}$$

for $\widehat{\Phi}$ to be odd $\widehat{\Omega}$ has to be even $|\widehat{\Omega}| = 0$. In component we have:

$$\widehat{\Omega} = \Omega_+ \otimes I + \Omega_- \otimes \sigma_1, \tag{19}$$

Ω_+ and Ω_- belong to $GSO(\pm)$ sectors.

It is obvious that two pure gauge configurations are related via a gauge transformation,

$$\begin{aligned}\widehat{\Phi}_1 &= \widehat{\Omega}_1^{-1}\widehat{Q}\widehat{\Omega}_1, &(20)\\ \widehat{\Phi}_2 &= \widehat{\Omega}_2^{-1}\widehat{Q}\widehat{\Omega}_2, &(21)\end{aligned}$$

$$\begin{aligned}\widehat{\Phi}_2 &= \widehat{\Omega}^{-1}(\widehat{\Phi}_1+\widehat{Q})\widehat{\Omega}, &(22)\\ \widehat{\Omega} &= \widehat{\Omega}_1^{-1}\widehat{\Omega}_2. &(23)\end{aligned}$$

A pure gauge solution in the $GSO(+)$ sector

$$\Phi_+ = \Omega_1^{-1}Q\Omega_1. \tag{24}$$

can be consider as a special pure gauge solution in the matrix case

$$\widehat{\Phi}_+ = \Omega_1^{-1}Q\Omega_1 \otimes \sigma_3. \tag{25}$$

This configuration is gauge equivalent to a given matrix pure gauge configuration $\widehat{\Phi}=\widehat{\Omega}^{-1}\widehat{Q}\widehat{\Omega}$, i.e.

$$\widehat{\Omega}^{-1}\widehat{Q}\widehat{\Omega} = \widehat{\Omega}_2^{-1}(\Omega_1^{-1}Q\Omega_1+Q)\otimes\sigma_3\widehat{\Omega}_2. \tag{26}$$

Indeed, from (26) we get

$$\Omega_{+2} = \Omega_1^{-1}\Omega_+, \quad \Omega_{-2} = \Omega_1^{-1}\Omega_-. \tag{27}$$

2.3 Perturbative Expansion in Matrix Notations

In this section we are going to find a solution of the equation of motion (17). The solution we will find as a series in some parameter λ i.e. let us suppose $\widehat{\Phi}$ to be a series in some λ

$$\widehat{\Phi}^{\lambda} = \sum_{n=0}^{\infty}\lambda^{n+1}\widehat{\phi}_n, \tag{28}$$

and put this expansion in the equation of motion (17). In the first order in λ we have

$$\widehat{Q}\widehat{\phi}_0 = 0. \tag{29}$$

We choose a solution to (29) as

$$\widehat{\phi}_0 = \widehat{Q}\widehat{\phi}, \tag{30}$$

where

$$\widehat{\phi} = \phi_+\otimes I + \phi_-\otimes\sigma_1, \tag{31}$$

ϕ_+ and ϕ_- are components of the gauge field $\widehat{\phi}$ and they belong to $GSO(+)$ and $GSO(-)$ sectors respectively. The Grassman parities of ϕ_+ and ϕ_- are opposite.

In the second order in λ we have

$$\widehat{Q}\widehat{\phi}_1 + \widehat{\phi}_0\star\widehat{\phi}_0 = 0. \tag{32}$$

For $\widehat{\phi}_0$ in the form (30) we get (also we used Leibnitz rule (11) for $\widehat{Q}$)

$$\widehat{Q}\widehat{\phi}_1 + \widehat{Q}\widehat{\phi}\star\widehat{Q}\widehat{\phi} = \widehat{Q}(\widehat{\phi}_1 - \widehat{Q}\widehat{\phi}\star\widehat{\phi}) = 0, \tag{33}$$

due to $|\widehat{\phi}|=0$ we get minus. The solution of equation (32) is

$$\widehat{\phi}_1=\widehat{Q}\widehat{\phi}\star\widehat{\phi}. \tag{34}$$

In this scheme we get

$$\widehat{\phi}_n=\widehat{Q}\widehat{\phi}\star\widehat{\phi}^n, \tag{35}$$

then $\widehat{\Phi}$ is

$$\widehat{\Phi}^\lambda=\sum_{n=0}^{\infty}\lambda^{n+1}\widehat{Q}\widehat{\phi}\star\widehat{\phi}^n=\lambda\widehat{Q}\widehat{\phi}\frac{1}{1-\lambda\widehat{\phi}}. \tag{36}$$

The perturbative solution has the pure gauge form (18). Indeed, let us introduce $\widehat{\Omega}=1-\lambda\widehat{\phi}$, then (36) is

$$\widehat{\Phi}^\lambda=-\widehat{Q}(1-\lambda\widehat{\phi})\star(1-\lambda\widehat{\phi})^{-1}=-\widehat{Q}\widehat{\Omega}\star\widehat{\Omega}^{-1}. \tag{37}$$

This expression can be written through $\phi_\pm$ as

$$\widehat{\Phi}^\lambda=(Q\phi_+\otimes\sigma_3+Q\phi_-\otimes i\sigma_2)\frac{1}{(1-\lambda\phi_+)^2-\lambda^2\phi_-^2}((1-\lambda\phi_+)\otimes I+\lambda\phi_-\otimes\sigma_1). \tag{38}$$

Picking out $GSO(+)$ and $GSO(-)$ sectors we get

$$\Phi_+^\lambda=\frac{\lambda}{2}Q(\phi_++\phi_-)\frac{1}{1-\lambda(\phi_++\phi_-)}+\frac{\lambda}{2}Q(\phi_+-\phi_-)\frac{1}{1-\lambda(\phi_+-\phi_-)}, \tag{39}$$

$$\Phi_-^\lambda=\frac{\lambda}{2}Q(\phi_++\phi_-)\frac{1}{1-\lambda(\phi_++\phi_-)}-\frac{\lambda}{2}Q(\phi_+-\phi_-)\frac{1}{1-\lambda(\phi_+-\phi_-)}. \tag{40}$$

This result is agree with [28].

3 Equivalence of BSZ and ABKM Theories

The action for the cubic NS string theory with $GSO(-)$ sector is presented in Section 2. In the non-polynomial theory the $GSO(-)$ sector can be added by the following way [30]. The field is an element of 2×2 matrix of the form

$$\widehat{G}=G_+\otimes I+G_-\otimes\sigma_1. \tag{41}$$

An equation of motion has the following form

$$\widehat{\eta}_0(\widehat{G}^{-1}\widehat{Q}\widehat{G})=0, \tag{42}$$

where $\widehat{\eta}_0\equiv\eta\otimes\sigma_3$.

Let $\mathfrak{A}$ be a set of matrix solutions of equation of motion (17) and $\mathfrak{B}$ is set of solutions (42).

Let us define a map g of $\mathfrak{B}$ to $\mathfrak{A}$ [32]

$$g:\widehat{G}\to\widehat{\Psi}\equiv g(\widehat{G})=\widehat{G}^{-1}\widehat{Q}\widehat{G}. \tag{43}$$

This map is correctly defined due to (42) ($\widehat{\Psi}$ is in the small Hilbert space [40]).

In order to $\widehat{\Psi} = g(\widehat{G})$ be a solution of equation of motion (17) it is necessary and sufficient to implement the Leibnitz rule for operator $\widehat{Q}$ (11). Let us note that G_+ is even and G_- is odd, i.e. G_+ and G_- have the different parities.

Let us define a map h of $\mathfrak{A}$ in $\mathfrak{B}$ as [32]

$$h : \widehat{\Psi} \to \widehat{G} \equiv h(\widehat{\Psi}) = e^{\widehat{P}\widehat{\Psi}}. \tag{44}$$

here $\widehat{P} \equiv P \otimes \sigma_3$, where P is nilpotent operator with respect to $\star$ defined in [32]

$$(P\Psi_1) \star (P\Psi_2) = 0, \tag{45}$$

and its anticommutator with Q is the identity

$$\{Q, P(z)\} = 1. \tag{46}$$

Since $\widehat{P}^2 = 0$ we have

$$e^{\widehat{P}\widehat{\Psi}} = 1 + \widehat{P}\widehat{\Psi}, \tag{47}$$

here 1 is an identity state $|I\rangle \otimes I$ with respect to $\star$.

In the components (44) reads

$$G_+ = 1 + P\Psi_+ = e^{P\Psi_+}, \quad G_- = P\Psi_-. \tag{48}$$

The maps g and h are connected nontrivially. Let us consider a composition $g \circ h$:

$$\begin{aligned} \widetilde{\widehat{\Psi}} &= (g \circ h)(\widehat{\Psi}) = g(h(\widehat{\Psi})) = (1 - \widehat{P}\widehat{\Psi})\widehat{Q}(1 + \widehat{P}\widehat{\Psi}) = (1 - \widehat{P}\widehat{\Psi})\widehat{Q}\widehat{P}\widehat{\Psi} \\ &= (1 - \widehat{P}\widehat{\Psi})(1 - \widehat{P}\widehat{Q})\widehat{\Psi} = (1 - \widehat{P}\widehat{Q} - \widehat{P}\widehat{\Psi})\widehat{\Psi} = \widehat{\Psi} - \widehat{P}(\widehat{Q}\widehat{\Psi} + \widehat{\Psi}^2) = \widehat{\Psi}, \end{aligned} \tag{49}$$

here, we used (46), then we used the equation of motion for $\widehat{\Psi}$ and the nilpotency of $\widehat{P}$ under the star product (45). So we have proved that $g \circ h = Id$ and $g(\mathfrak{B}) = \mathfrak{A}$ i.e. an arbitrary classical solution in cubic theory can be represent in pure-gauge form.

Now let us consider a composition $h \circ g$:

$$\begin{aligned} \widetilde{\widehat{G}} &= (h \circ g)(\widehat{G}) = h(g(\widehat{G})) = e^{\widehat{P}\widehat{G}^{-1}\widehat{Q}\widehat{G}} = 1 + \widehat{P}\widehat{G}^{-1}\widehat{Q}\widehat{G} = 1 - \widehat{P}\widehat{Q}\widehat{G}^{-1} \cdot \widehat{G} \\ &= 1 - (1 - \widehat{Q}\widehat{P})\widehat{G}^{-1} \cdot \widehat{G} = 1 - 1 + \widehat{Q}\widehat{P}\widehat{G}^{-1} \cdot \widehat{G} = \widehat{Q}\widehat{P}\widehat{G}^{-1} \cdot \widehat{G}. \end{aligned} \tag{50}$$

For an arbitrary $\widehat{G} \in \mathfrak{B}$ introduce the following parametrization [32]

$$\widehat{G} = \frac{1}{1 - \widehat{\Phi}}. \tag{51}$$

The element $\widehat{\Psi} = g(\widehat{G}) \in \mathfrak{A}$ takes the form

$$\widehat{\Psi} = \widehat{G}^{-1}\widehat{Q}\widehat{G} = -\widehat{Q}\widehat{G}^{-1}\widehat{G} = \widehat{Q}\widehat{\Phi}\frac{1}{1 - \widehat{\Phi}}. \tag{52}$$

Here it is used that $\widehat{G}$ is even, the Leibnitz rule is used, at the same time in was important, that the parities of G_+ and G_- are opposite and $\sigma_2 I = I\sigma_2$, $\sigma_3\sigma_1 = -\sigma_1\sigma_3$. Also we used that P changes the parity of field.

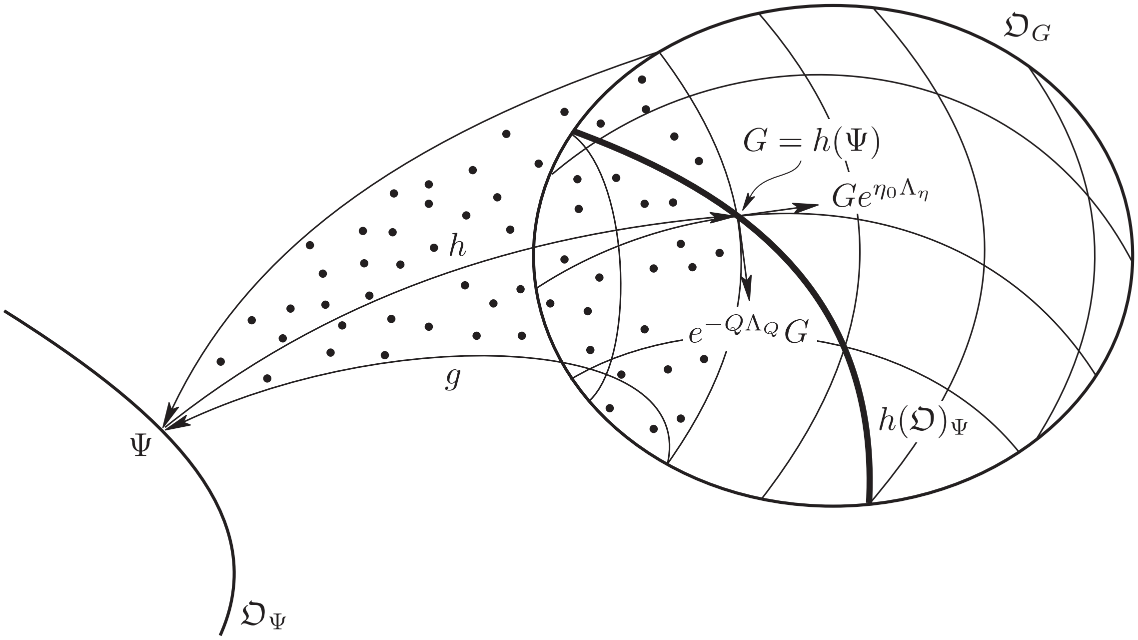

Figure 1: Maps h and g. Here hats are omitted for simplicity.

Then we use the parametrization (51) for (50)

$$\widetilde{\widehat{G}} = \widehat{Q}\widehat{P}\widehat{G}^{-1} \cdot \widehat{G} = \widehat{Q}\widehat{P}(1-\widehat{\Phi}) \cdot \frac{1}{1-\widehat{\Phi}} = \frac{1}{1-\widehat{\Phi}} - \widehat{Q}\widehat{P}\widehat{\Phi}\frac{1}{1-\widehat{\Phi}} = (1-\widehat{Q}(\widehat{P}\widehat{\Phi}))\widehat{G}, \quad (53)$$

here we use

$$\widehat{Q}\widehat{P}I = I. \quad (54)$$

Let us rewrite (53) as

$$\widetilde{\widehat{G}} = e^{-\widehat{Q}(\widehat{P}\widehat{\Phi})}\widehat{G}. \quad (55)$$

It is the gauge transformation

$$\widetilde{\widehat{G}} = e^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}, \quad (56)$$

with a gauge parameter $\widehat{\Lambda}_{\widehat{Q}} = \widehat{P}\widehat{\Phi}$, $\widehat{\Lambda}_{\widehat{\eta}} = 0$.

So $(h \circ g)(\widehat{G})$ belongs to a gauge orbit $\mathfrak{O}_{\widehat{G}} = \{\widetilde{\widehat{G}} : \widetilde{\widehat{G}} = e^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}\}$ of the initial field $\widehat{G}$. In the components (55) reads

$$\begin{aligned} \widetilde{G}_+ &= e^{-Q\Lambda_+}G_+ - Q\Lambda_- G_-, \\ \widetilde{G}_- &= e^{-Q\Lambda_+}G_- - Q\Lambda_- G_+, \end{aligned} \quad (57)$$

$$\widehat{\Lambda}_Q = \Lambda_+ \otimes \sigma_3 + \Lambda_- \otimes i\sigma_2, \quad (58)$$

where $\Lambda_+ = P\Phi_+, \quad \Lambda_- = P\Phi_-$.

In the terms of gauge orbits the maps g and h can be describe more clearly.

Let $\widehat{\Psi}$ be an arbitrary field of $\mathfrak{A}$ and $\widehat{G} = h(\widehat{\Psi})$. Let us consider an image of the orbit $\mathfrak{O}_{\widehat{\Psi}} = \{\widetilde{\widehat{\Psi}} : \widetilde{\widehat{\Psi}} = e^{-\widehat{\Lambda}}(\widehat{\Psi} + \widehat{Q})e^{\widehat{\Lambda}}\}$ by the map h: $h(\mathfrak{O}_{\widehat{\Psi}}) = \{\widetilde{\widehat{G}} : \widetilde{\widehat{G}} = h(\widetilde{\widehat{\Psi}})\}$. The direct calculation gives:

$$
\begin{aligned}
\widetilde{\widehat{G}} &= 1 + \widehat{P}\widetilde{\widehat{\Psi}} = 1 + \widehat{P}(e^{-\widehat{\Lambda}}(\widehat{\Psi} + \widehat{Q})e^{\widehat{\Lambda}}) = 1 + \widehat{P}(-\widehat{Q}e^{-\widehat{\Lambda}} + e^{-\widehat{\Lambda}}\widehat{\Psi})e^{\widehat{\Lambda}} \\
&= (\widehat{Q}(Pe^{-\widehat{\Lambda}}) + \widehat{P}e^{-\widehat{\Lambda}}\widehat{\Psi})e^{\widehat{\Lambda}} = \widehat{Q}(\widehat{P}e^{-\widehat{\Lambda}})(1 + \widehat{P}\widehat{\Psi})e^{\widehat{\Lambda}} = \widehat{Q}(\widehat{P}e^{-\widehat{Q}\widehat{P}\widehat{\Lambda}})\widehat{G}e^{\widehat{\Lambda}} \\
&= e^{-\widehat{Q}\widehat{P}\widehat{\Lambda}}\widehat{G}e^{\widehat{\Lambda}} = e^{-\widehat{Q}\widehat{P}\widehat{\Lambda}}\widehat{G}e^{\widehat{\eta}_0\widehat{\xi}\widehat{\Lambda}},
\end{aligned}
\tag{59}
$$

i.e. $h(\mathfrak{O}_{\widehat{\Psi}})$ is suborbit of the field $\widehat{G} = h(\widehat{\Psi})$, due to a special choose of gauge parameter $\widehat{\Lambda}_{\widehat{Q}}$, $\widehat{\Lambda}_{\widehat{\eta}}$ or $h(\mathfrak{O}_{\widehat{\Psi}}) \subset \mathfrak{O}_{\widehat{G}}$.

Let $\widehat{G}$ be an arbitrary field of $\mathfrak{B}$ and $\widehat{\Psi} = g(\widehat{G})$. Let us consider an image of the orbit $\mathfrak{O}_{\widehat{G}}$ by the map g: $g(\mathfrak{O}_{\widehat{G}}) = \{\widetilde{\widehat{\Psi}} = g(\widetilde{\widehat{G}}) : \widetilde{\widehat{G}} \in \mathfrak{O}_{\widehat{G}}\}$:

$$
\begin{aligned}
\widetilde{\widehat{\Psi}} &= \widetilde{\widehat{G}}^{-1}\widehat{Q}\widetilde{\widehat{G}} = e^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\widehat{G}^{-1}e^{\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{Q}(e^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}) \\
&= e^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\widehat{G}^{-1}((\widehat{Q}\widehat{G})e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}} + \widehat{G}\widehat{Q}e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}) = e^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}(\widehat{\Psi} + \widehat{Q})e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}},
\end{aligned}
\tag{60}
$$

since $\widehat{\Lambda}_{\widehat{\eta}}$ is arbitrary, then $g(\mathfrak{O}_{\widehat{G}}) = \mathfrak{O}_{\widehat{\Psi}}$. Note that, if $h(\widehat{\Psi}') \in \mathfrak{O}_{h(\widehat{\Psi})}$, then $\widehat{\Psi}' \in \mathfrak{O}_{\widehat{\Psi}}$. Indeed, by virtue of $g \circ h = Id$ it is possible to rewrite $\widehat{\Psi}' = g(h(\widehat{\Psi}'))$, and since $g(\mathfrak{O}_{\widehat{G}}) = \mathfrak{O}_{\widehat{\Psi}}$, then $h(\widehat{\Psi}') \in \mathfrak{O}_{h(\widehat{\Psi})}$.

So we can see that the maps g and h could be constrict to the maps orbits:

$$
h : \mathfrak{O}_{\widehat{\Psi}} \to \mathfrak{O}_{\widehat{G}}, \quad g : \mathfrak{O}_{\widehat{G}} \to \mathfrak{O}_{\widehat{\Psi}}
$$

At the same time the image $\mathfrak{O}_{\widehat{\Psi}}$ in $\mathfrak{O}_{\widehat{G}}$ is suborbit (62). The image $\mathfrak{O}_{\widehat{G}} = \{\widetilde{\widehat{G}} : \widetilde{\widehat{G}} = e^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\}$ is all orbit $\mathfrak{O}_{\widehat{\Psi}}$. All elements $\mathfrak{O}_{\widehat{G}}$ with different $\widehat{\Lambda}_{\widehat{Q}}$ are mapped in one element $\mathfrak{O}_{\widehat{\Psi}}$ (see (60)). Bounded on $h(\mathfrak{O}_{\widehat{\Psi}})$ mapping g becomes invertible: $h \circ g|_{h(\mathfrak{O}_{\widehat{\Psi}})} = Id$. The composition $h \circ g$ gives in the orbit $\mathfrak{O}_{\widehat{G}}$ a special section (50). See figure 1.

4 Perturbative Expansion of Pure Gauge Configurations

4.1 Initial data and Formal Perturbative Expansion in Components

Here we choose ϕ_+ and ϕ_- in the following form [28]

$$
\begin{aligned}
\phi_+ &= B_1^L c_1|0\rangle, &(61)\\
\phi_- &= B_1^L \gamma_{\frac{1}{2}}|0\rangle. &(62)
\end{aligned}
$$

Then Φ^λ_+ and Φ^λ_- will have the form

$$\begin{aligned}
\Phi^\lambda_+ &= \sum_{n=0}^{\infty} \lambda^{n+1}\phi'_n, && (63)\\
\phi'_0 &= \Big(-K^R_1 c_1 - B^R_1(c_0c_1 + \gamma^2_{1/2})\Big)\,|0\rangle, && (64)\\
\phi'_n &= c_1|0\rangle \star |n\rangle \star K^L_1 B^L_1 c_1|0\rangle + \gamma_{\frac{1}{2}}|0\rangle \star |n\rangle \star K^L_1 B^L_1 \gamma_{\frac{1}{2}}|0\rangle,\;\; n>0, && (65)
\end{aligned}$$

$$\begin{aligned}
\Phi^\lambda_- &= \sum_{n=0}^{\infty} \lambda^{n+1}\psi'_n, && (66)\\
\psi'_0 &= \left(-K^R_1 \gamma_{\frac{1}{2}} + B^R_1(c_1\gamma_{-\frac{1}{2}} - \frac{1}{2}c_0\gamma_{\frac{1}{2}})\right)|0\rangle, && (67)\\
\psi'_n &= \gamma_{\frac{1}{2}}|0\rangle \star |n\rangle \star K^L_1 B^L_1 c_1|0\rangle + c_1|0\rangle \star |n\rangle \star K^L_1 B^L_1 \gamma_{\frac{1}{2}}|0\rangle\,,\;\; n>0. && (68)
\end{aligned}$$

4.2 $\lambda = 1$ limit

In this section we examine $\lambda = 1$ limit of the pure gauge solutions (38). It is known that this is a singular point for the pure gauge solution [1, 3].

We consider for the transparency the pure $GSO(+)$ sector and the equation of motion for the string field Φ_+ is

$$Q\Phi_+ + \Phi_+ \star \Phi_+ = 0. \tag{69}$$

We start with the pure gauge solution to (69) given by formulae (63) – (68) with $\lambda < 1$ and initial date $\phi_- = 0$. The explicit form of this solution is

$$\Phi_+(\lambda) = \sum_{n=0}^{\infty} \lambda^{n+1}\varphi'_n + \lambda\Gamma, \qquad |\lambda| < 1, \tag{70}$$

where

$$\begin{aligned}
\Gamma &= B^L_1\gamma^2_{1/2}|0\rangle,\\
\varphi'_0 &= -\left(K^R_1 c_1 + B^R_1 c_0 c_1\right)|0\rangle,\\
\varphi'_n &= c_1|0\rangle \star |n\rangle \star K^L_1 B^L_1 c_1|0\rangle \;\; n>0.
\end{aligned} \tag{71}$$

Let us take just a partial sum of the infinite series (70)

$$\Phi^N_+(\lambda) = \sum_{n=0}^{N-1} \lambda^{n+1}\varphi'_n + \lambda\Gamma, \tag{72}$$

and check a validity of the equation of motion (69) in a weak sense on the states φ_m [1]

$$\langle\langle \varphi_m, Q\Phi^N_+(\lambda) + \Phi^N_+(\lambda) \star \Phi^N_+(\lambda)\rangle\rangle, \tag{73}$$

[1]Here $\langle\langle ... \rangle\rangle = \langle Y_{-2} ... \rangle$

where

$$\varphi_m = \frac{2}{\pi} c_1|0\rangle \star |m\rangle \star B_1^L c_1|0\rangle. \tag{74}$$

We use correlators [33] collected in the table below

$$\begin{aligned}
\langle\langle \varphi_m, Q\varphi_n\rangle\rangle &= -\frac{m+n+2}{\pi^2},\\
\langle\langle \varphi_m, Q\Gamma\rangle\rangle &= \frac{1}{\pi^2},\\
\langle\langle \Gamma, Q\Gamma\rangle\rangle &= 0,\\
\langle\langle \varphi_k, \varphi_m \star \varphi_n\rangle\rangle &= 0,\\
\langle\langle \Gamma, \varphi_m \star \varphi_n\rangle\rangle &= \frac{m+n+3}{2\pi^2},\\
\langle\langle \Gamma, \Gamma \star \varphi_n\rangle\rangle &= 0,\\
\langle\langle \Gamma, \Gamma \star \Gamma\rangle\rangle &= 0.
\end{aligned} \tag{75}$$

We get

$$\langle\langle \varphi_m, Q\Phi_+^N(\lambda) + \Phi_+^N(\lambda) \star \Phi_+^N(\lambda)\rangle\rangle = \frac{\lambda^{N+1}}{\pi^2}. \tag{76}$$

Taking the limit $N \to \infty$ for $\lambda < 1$ we have for an arbitrary m

$$\langle\langle \varphi_m, Q\Phi_+(\lambda) + \Phi_+(\lambda) \star \Phi_+(\lambda)\rangle\rangle = 0, \tag{77}$$

in other words for $\lambda < 1$ the field $\Phi_+(\lambda)$ solves the E.O.M. when contracted with states from the subspase $\mathcal{L}(\{\varphi_m\})$ spanned by φ_m. This fact is natural for the solution obtained by the iteration procedure. It is interesting to note that if we consider the validity of the equation of motion on the subspace spanned by φ'_m we get that on this subspace the equation of motion are satisfied for any λ

$$\langle\langle \varphi'_m, Q\Phi_+(\lambda) + \Phi_+(\lambda) \star \Phi_+(\lambda)\rangle\rangle = 0. \tag{78}$$

From equation (76) one sees that for $\lambda = 1$ the string field $\Phi_+ \equiv \Phi_+(1)$ does not solve the equation of motion (69) in the week sense on $\mathcal{L}(\{\varphi_m\})$

$$\langle\langle \varphi_m, Q\Phi_+(1) + \Phi_+(1) \star \Phi_+(1)\rangle\rangle = \frac{1}{\pi^2}. \tag{79}$$

Let us remind that in the case of boson string to ensure the equation of motion in the sense (79) extra terms have been added to Φ_{bos}^N [41] and these extra terms provide the validity of the Sen conjecture [1, 3].

Following Erler [33] we can try to add to $\Phi_+^N \equiv \sum_{n=0}^{N-1} \varphi'_n + \Gamma$ two extra terms

$$\Phi_+^N(c_1, c_2) = \Phi_+^N + c_1\varphi_N + c_2\varphi'_N \tag{80}$$

and find c_1 and c_2 from a requirement of the validity of the equation of motion in the weak sense,

$$\langle\langle \varphi_m, Q\Phi_+^N(c_1,c_2) + \Phi_+^N(c_1,c_2) \star \Phi_+^N(c_1,c_2)\rangle\rangle = 0. \tag{81}$$

Simple calculations based on (75) show that $c_1 = -1$ and c_2 is arbitrary. Indeed,

$$\begin{aligned}
\langle\langle \varphi_m, Q\Phi_+^N(c_1,c_2)\rangle\rangle &= -\frac{N-1}{\pi^2} - c_1\frac{m+N+2}{\pi^2} - c_2\frac{1}{\pi^2},\\
\langle\langle \varphi_m, \Phi_+^N(c_1,c_2) \star \Phi_+^N(c_1,c_2)\rangle\rangle &= \frac{N}{\pi^2} + c_1\frac{m+N+3}{\pi^2} + c_2\frac{1}{\pi^2}
\end{aligned} \tag{82}$$

and we see that

$$\begin{aligned}
&\langle\langle\varphi_m, Q\Phi^N_+(c_1,c_2)+\Phi^N_+(c_1,c_2)\star\Phi^N_+(c_1,c_2)\rangle\rangle \\
=\;& -\frac{N-1}{\pi^2}-c_1\frac{m+N+2}{\pi^2}-c_2\frac{1}{\pi^2}+\frac{N}{\pi^2}+c_1\frac{m+N+3}{\pi^2}+c_2\frac{1}{\pi^2} \\
=\;& \frac{1}{\pi^2}+c_1\frac{1}{\pi^2}
\end{aligned} \tag{83}$$

is equal to zero for $c_1=-1$.

Let us add to our subspace $\mathcal{L}(\{\varphi_m\})$ a vector Γ and consider the requirement of the validity of the equation of motion also on this vector

$$\langle\langle\Gamma, Q\Phi^N_+(-1,c_2)+\Phi^N_+(-1,c_2)\star\Phi^N_+(-1,c_2)\rangle\rangle=0. \tag{84}$$

We have

$$\langle\langle\Gamma, Q\Phi^N_+(-1,c_2)+\Phi^N_+(-1,c_2)\star\Phi^N_+(-1,c_2)\rangle\rangle=-\frac{1}{\pi^2}+\frac{3}{2\pi^2}-c_2\frac{1}{\pi^2}. \tag{85}$$

and we see that the L.H.S. of (85) is zero for $c_2=1/2$. So

$$\Phi^N_+(-1,1/2)=\sum_{n=0}^{N-1}\varphi'_n+\Gamma-\varphi_N+\frac{1}{2}\varphi'_N. \tag{86}$$

It is interesting to note that $c_1=-1, c_2=1/2$ provide the validity of the equation of motion being contracting with $\Phi^N_+(-1,1/2)$

$$\langle\langle\Phi^N_+(-1,1/2), Q\Phi^N_+(-1,1/2)\;+\;\Phi^N_+(-1,1/2)\star\Phi^N_+(-1,1/2)\rangle\rangle=0. \tag{87}$$

Therefore, we see that just the requirement of the validity of E.O.M. "terms by terms" at the point $\lambda=1$ forces one to add two extra terms to Φ^N_+. A necessity of these extra terms have been advocated in [33] to provide the Sen conjecture.

5 Conclusion

In this article a singular limit of the pure gauge solution is discussed. We propose a simple recept to deal with a singularity problem and on the example of cubic SSFT show that it gives the same answer as the requirement to get a desirable value of the action [33] (see the discussion of the same question for the case with $GSO(-)$ sector in [41])

The equivalence of the solutions of the equation of motion in the cubic fermionic string field theory [27] and that of the non-polynomial fermionic string field theory [30] including the $GSO(-)$ sectors is discussed using the matrix representations of both theories. However the singularity problem recall that a formal gauge equivalence of two theories needs a rather delicate studies.

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